

EQUIVARIANT BRANES

ANDRÉS VIÑA

ABSTRACT. Given a Calabi-Yau manifold X acted by a group G and considering the B -branes on X as objects in the derived category of coherent sheaves, we give a definition of equivariant branes, which generalizes the concept of equivariant sheaves. We also propose a definition of equivariant charge of an equivariant brane. The spaces of strings joining the branes \mathcal{F} and \mathcal{G} , are the groups $Ext^i(\mathcal{F}, \mathcal{G})$. We prove that the spaces of strings between two G -equivariant branes support representations of G . Thus, these spaces can be decomposed in direct sum of invariant spaces for the G -action. We show some particular decompositions, when X is a toric variety and when X is a flag manifold of a semisimple Lie group.

MSC 2010: 57S20, 55N91, 14F05

1. INTRODUCTION

As it is known, a D -brane of type B in a Calabi-Yau manifold X is an object of the derived category of coherent sheaves on X [1, 2, 3, 11, 21, 28]. In this note we will consider such objects in manifolds acted by a Lie group G .

Given a G -manifold X , some objects related with X admit an “equivariant” version. For example, the equivariant vector bundles on X are vector bundles equipped with a structure compatible with the G -action on the base. In the same way, it is natural to consider “equivariant” B -branes on X . In this article, we deal with equivariant branes on a G -manifold, with the spaces of open strings connecting them and we will relate these spaces with representations of the group G .

Henceforth, the space X will be a Kähler G -manifold, and we put \mathcal{O}_X for the corresponding structure sheaf. By $\mathbf{D}(\mathcal{O}_X)$ we denote the bounded derived category of coherent \mathcal{O}_X -modules [20]. In the context mentioned above, given the B -branes \mathcal{F} and \mathcal{G} , that are objects of $\mathbf{D}(\mathcal{O}_X)$, an open string between \mathcal{F} and \mathcal{G} is an element of the Ext group $Ext_{\mathcal{O}_X}^i(\mathcal{F}, \mathcal{G})$ [1, 30], where $i + \text{ghost number of } \mathcal{G} -$

Key words and phrases. B -branes, equivariant cohomology, derived categories of sheaves.

ghost number of \mathcal{F} can be considered as the ghost number of the corresponding strings.

We denote by $\mu : G \times X \rightarrow X$ an *analytic* action of a reductive Lie group G on X . Essentially, a G -equivariant structure on the \mathcal{O}_X -module \mathcal{H} is given by a family $\{\alpha_{g,x}\}$ of isomorphisms between the stalks

$$\alpha_{g,x} : \mathcal{H}_x \rightarrow \mathcal{H}_{\mu(g,x)}, \quad \text{for all } g \in G, x \in X$$

compatible with the multiplication in G .

For a precise definition, we introduce the map $b : (g, x) \in G \times X \mapsto x \in X$. A G -equivariant \mathcal{O}_X -module is a pair (\mathcal{H}, α) , where \mathcal{H} is an \mathcal{O}_X -module and α is an isomorphism

$$(1.1) \quad \alpha : b^* \mathcal{H} \rightarrow \mu^* \mathcal{H}$$

of $\mathcal{O}_{G \times X}$ -modules, where b^* and μ^* are the functors inverse image defined by the respective maps [13, page 136]. Furthermore, α must satisfy the cocycle condition (see [6, page 2] and equation (2.2) below).

The same definition of G -equivariance is applicable to an object \mathcal{A} of the derived category $\mathbf{D}(\mathcal{O}_X)$, now b^* and μ^* are functors from $\mathbf{D}(\mathcal{O}_X)$ to the derived category of $\mathcal{O}_{G \times X}$ -modules (see Subsection 2.1).

We will put $j : S \hookrightarrow X$ for the inclusion of an open subset S of X , and $j_!$ will denote the corresponding functor direct image with compact support [20, page 103].

In Section 2, we will prove the following theorems.

Theorem 1. *If (\mathcal{H}, α) is a G -equivariant coherent sheaf on X and S is a G -invariant open subset of X , then the isomorphism α determines a representation of G on $St^i(j_!(\mathcal{O}_S), \mathcal{H})$, all i . In particular, each space $St^i(\mathcal{O}_X, \mathcal{H})$ carries a representation of G induced by α .*

Theorem 2. *If (\mathcal{G}, β) and (\mathcal{F}, γ) are G -equivariant objects of the category $\mathbf{D}(\mathcal{O}_X)$, then the isomorphisms β and γ determine a representation of G on $St^i(\mathcal{F}, \mathcal{G})$, in a natural way.*

In the particular case that \mathcal{F} and \mathcal{G} are the sheaves of sections of G -equivariant vector bundles, the action $g \in G$ on a morphism

$$\Phi \in St^0(\mathcal{F}, \mathcal{G}) = Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

is given by the natural representation $(g \cdot \Phi)(-) = g\Phi(g^{-1}(-))$.

The decomposition of the representations of compact groups in direct sum of irreducible representations permits to classify the elements of a given space $St^i(\mathcal{F}, \mathcal{G})$ in subspaces, which can be labelled by the characters of the corresponding irreducible representations. Thus, we have the following theorem.

Theorem 3. *Let G be a compact group G . If \mathcal{F} and \mathcal{G} are G -equivariant objects of $\mathbf{D}(X)$ then for each i ,*

$$(1.2) \quad St^i(\mathcal{F}, \mathcal{G}) \simeq \bigoplus_A n_A A,$$

where the sum runs over a complete set of pairwise nonisomorphic representations of G , n_A is a natural number and $n_A A$ is the direct sum of n_A summands of the irreducible representation A .

In Section 2, we give an equivariant version of the charge of a G -equivariant brane, that coincides with the usual one, when the group G is trivial (see Subsection 2.4). This equivariant charge can be evaluated using the localization formulas in equivariant cohomology [4].

Examples manifolds in which a group action is an important ingredient of its structure are the coadjoint orbits of a Lie group and the toric manifolds. In Section 3, we will show the form which Theorem 3 adopts in some examples of pairs of \mathcal{O}_X -modules, when X is a toric manifold and when X is a flag manifold of a semisimple group.

Notations. Besides the already introduced notations, we also use the following:

The category of complex vector spaces will be denoted by \mathfrak{Vect} , and we let $D(\mathfrak{Vect})$ for the corresponding bounded derived category.

Given a locally compact space Z , if \mathcal{R} a sheaf of \mathbb{C} -algebras on Z , the bounded derived category of sheaves on Z which are \mathcal{R} -modules is denoted by $D(\mathcal{R})$. As usual, $\Gamma(Z, \cdot)$ will be the functor global sections and we put

$$R\Gamma(Z, \cdot) : D(\mathcal{R}) \rightarrow D(\mathfrak{Vect})$$

for its derived [20, 30]. The composition of this functor with the cohomology functor H^i is denoted by $R^i\Gamma(Z, \cdot)$

$$R^i\Gamma(Z, \cdot) : D(\mathcal{R}) \rightarrow \mathfrak{Vect}.$$

If $f : Y \rightarrow Z$ is a continuous map,

$$Rf_* : D(f^*\mathcal{R}) \rightarrow D(\mathcal{R})$$

will denote the derived functor of the direct image functor f_* .

In general, if Z is a ringed space the structure sheaf will be denoted by \mathcal{O}_Z .

Acknowledgement. I thank to Diego Rodríguez-Gómez for pointing the reference [1] out to me.

2. PROOFS OF THE RESULTS

2.1. The cocycle condition. To formulate the cocycle condition, above mentioned, we introduce the following notations

$$m : G \times G \rightarrow G, \quad m(g_1, g_2) = g_1 g_2.$$

$$p : G \times G \times X \rightarrow G \times X, \quad p(g_1, g_2, x) = (g_2, x).$$

Thus, one has the maps p , $m \times 1_X$ and $1_G \times \mu$ from $G \times G \times X$ to $G \times X$ and the corresponding functors

$$(2.1) \quad \begin{array}{ccccc} \mathbf{D}(\mathcal{O}_X) & \xrightleftharpoons[\mu^*]{b^*} & D(\mathcal{O}_{G \times X}) & \xrightarrow[p^*]{(m \times 1_X)^*} & D(\mathcal{O}_{G \times G \times X}), \\ & & & \xleftarrow[(1_G \times \mu)^*]{} & \end{array}$$

where an asterisk as superscript is used for denoting the inverse image functor between the corresponding derived categories. The equalities

$$b \circ (m \times 1_X) = b \circ p, \quad b \circ (1_G \times \mu) = \mu \circ p, \quad \mu \circ (1_G \times \mu) = \mu \circ (m \times 1_X)$$

give rise to equalities between the respective compositions of the functors in (2.1).

Given an object \mathcal{A} of the category $\mathbf{D}(\mathcal{O}_X)$, an isomorphism $\alpha : b^* \mathcal{A} \rightarrow \mu^* \mathcal{A}$ satisfies the cocycle condition if

$$(2.2) \quad (m \times 1_X)^*(\alpha) = (1_G \times \mu)^*(\alpha) \circ p^*(\alpha).$$

In this case, we say that the pair (\mathcal{A}, α) is an G -equivariant object.

Both sides of equation (2.2) are isomorphisms between two objects on the derived category of $\mathcal{O}_{G \times G \times X}$ -modules; more precisely, between the objects $d_0^* \mathcal{A}$ and $d^* \mathcal{A}$, where $d, d_0 : G \times G \times X \rightarrow X$ are defined by $d_0(g_1, g_2, x) = x$ and $d(g_1, g_2, x) = (g_1 g_2)x$. In other words, the cocycle condition means the commutativity of the following triangle

$$(2.3) \quad \begin{array}{ccc} \mathcal{Z}_1 & \xrightarrow{p^*(\alpha)} & \mathcal{Z}_2 \\ & \searrow (m \times 1_X)^*(\alpha) & \swarrow (1_G \times \mu)^*(\alpha) \\ & \mathcal{Z}_3 & \end{array}$$

where

$$\begin{aligned} \mathcal{Z}_1 &:= p^* b^*(\mathcal{A}) = (m \times 1_X)^* b^*(\mathcal{A}), \quad \mathcal{Z}_2 := p^* \mu^*(\mathcal{A}) = (1_G \times \mu)^* b^*(\mathcal{A}) \\ \mathcal{Z}_3 &:= (m \times 1_X)^* \mu^*(\mathcal{A}) = (1_G \times \mu)^* \mu^*(\mathcal{A}). \end{aligned}$$

2.2. Equivariant sheaves. Now we suppose that the G -equivariant object \mathcal{A} is a coherent sheaf \mathcal{H} on X . One can define the category $\text{Coh}^G(X)$, whose objects are the G -equivariant coherent sheaves on X . If (\mathcal{H}', α') and (\mathcal{H}, α) are objects in this category, a morphism in $\text{Coh}^G(X)$ from (\mathcal{H}', α') to (\mathcal{H}, α) is a sheaf morphism $f : \mathcal{H}' \rightarrow \mathcal{H}$ such that $\alpha b^*(f) = \mu^*(f)\alpha'$.

Given an open subset $U \subset X$ and $g \in G$, we put $U_g := \{g\} \times U \subset G \times X$. If (\mathcal{H}, α) is an object of $\text{Coh}^G(X)$, the restriction to U_g of the morphism of sheaves α is denoted $\alpha|_{U_g}$

$$(2.4) \quad \alpha|_{U_g} : b^*\mathcal{H}(U_g) = \mathcal{H}(U) \longrightarrow \mu^*\mathcal{H}(U_g) = \mathcal{H}(gU).$$

Hence, one has the following proposition.

Proposition 4. $\alpha|_{U_g}$ determines an isomorphism of complex vector spaces

$$(2.5) \quad \mathcal{H}(U) \xrightarrow{\sim} \mathcal{H}(gU)$$

The image of $\sigma_U \in \mathcal{H}(U)$ will be denoted $g \cdot \sigma_U$. Thus, the element $g \in G$ determines an isomorphism

$$(2.6) \quad \alpha_{g,x} : \mathcal{H}_x \rightarrow \mathcal{H}_{gx},$$

for any $x \in X$.

Lemma 5. Let g, h be elements of G and U an open set of X , then

$$\alpha|_{(gU)_h} \circ \alpha|_{U_g} = \alpha|_{U_{hg}}.$$

Proof. We consider the commutative triangle (2.3) when \mathcal{A} is the sheaf \mathcal{H} and we restrict this triangle to $\{h\} \times U_g \subset G \times G \times X$. The restriction of $(m \times 1_X)^*(\alpha)$ is the morphism

$$\mathcal{Z}_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow \mathcal{Z}_3(\{h\} \times U_g) = \mathcal{H}((hg)U)$$

induced by α . Thus, by (2.4), the mentioned restriction is $\alpha|_{U_{hg}}$.

The restriction of $p^*(\alpha)$ to $\{h\} \times U_g$

$$\mathcal{Z}_1(\{h\} \times U_g) = \mathcal{H}(U) \longrightarrow \mathcal{Z}_2(\{h\} \times U_g) = \mathcal{H}(gU)$$

is (2.4).

Finally, we consider the restriction of $(1_G \times \mu)^*(\alpha)$. It is the morphism

$$\mathcal{Z}_2(\{h\} \times U_g) = \mathcal{H}(gU) \longrightarrow \mathcal{Z}_2(\{h\} \times U_g) = \mathcal{H}(h(gU))$$

induced by α , and according to (2.4) it is $\alpha|_{(gU)_h}$. Then the lemma follows from the commutativity of (2.3). \square

As a direct consequence of Lemma

$$(2.7) \quad \alpha_{h,gx} \circ \alpha_{g,x} = \alpha_{hg,x}$$

Since $gX = X$, $\alpha|_{X_g}$ is an automorphism of the vector space $\mathcal{H}(X)$, from the Lemma, one deduces

$$(2.8) \quad \alpha|_{X_h} \circ \alpha|_{X_g} = \alpha|_{X_{hg}}.$$

The following proposition is a consequence from (2.8).

Proposition 6. *The automorphisms $\{\rho_g := \alpha|_{X_g}\}_g$ form a representation of G in the vector space $\mathcal{H}(X)$.*

Lemma 7. *Given (\mathcal{H}, α) an object of $\text{Coh}^G(X)$, there is an resolution of (\mathcal{H}, α) in $\text{Coh}^G(X)$ consisting of injective \mathcal{O}_X -modules.*

Proof. Given a point $x \in X$, \mathcal{H}_x is a \mathbb{C} -vector space, so is an injective \mathbb{Z} -module (i.e. a divisible abelian group). We set $I_x := \text{Hom}_{\mathbb{Z}}(\mathcal{O}_x, \mathcal{H}_x)$, where \mathcal{O}_x is the stalk of the sheaf \mathcal{O}_X at the point x . Then the inclusion $\mathcal{H}_x \hookrightarrow I_x$ is an embedding of the \mathcal{O}_x -module \mathcal{H}_x in an injective \mathcal{O}_x -module (see [24, III.7], [27, page 123]).

By means of the I_x one can construct an injective \mathcal{O}_X -module \mathcal{J} in which \mathcal{H} can be embedded.

$$\mathcal{J} = \prod_{x \in X} (j_x)_*(I_x),$$

where $j_x : \{x\} \hookrightarrow X$ (see [17, page 207]).

As it is well-known, $\mathcal{H} \xrightarrow{i} \mathcal{J}$ is the 0th-term of an injective resolution of the \mathcal{O}_X -module \mathcal{H} . As \mathcal{H} is G -equivariant, by (2.6), for each $g \in G$, there is an isomorphism of \mathbb{Z} -modules $\mathcal{H}_x \rightarrow \mathcal{H}_{gx}$. Since the G -action on X is analytic, one has an isomorphism $\mathcal{O}_{gx} \rightarrow \mathcal{O}_x$. So, we have an isomorphism $I_x \rightarrow I_{gx}$. Thus, for any open subset $U \subset X$, there is an isomorphism of \mathbb{C} -vector spaces $\mathcal{J}(U) \rightarrow \mathcal{J}(gU)$, making commutative the following diagram, where the horizontal sequences are exact

$$(2.9) \quad \begin{array}{ccccc} 0 & \longrightarrow & \mathcal{H}(U) & \xrightarrow{i_U} & \mathcal{J}(U) \\ & & \alpha|_{U_g} \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}(gU) & \xrightarrow{i_{gU}} & \mathcal{J}(gU) \end{array}$$

The isomorphisms $\mathcal{J}(U) \rightarrow \mathcal{J}(gU)$ give rise to an isomorphism of \mathcal{O}_X -modules, $\alpha^0 : b^* \mathcal{J} \rightarrow \mu^* \mathcal{J}$.

From (2.7), it follows the equality of $\mathcal{J}(U) \rightarrow \mathcal{J}(gU) \rightarrow \mathcal{J}(hgU)$ and $\mathcal{J}(U) \rightarrow \mathcal{J}((hg)U)$. That is, α^0 satisfies the cocycle condition. Hence, \mathcal{J} is an object of $\text{Coh}^G(X)$. By the commutativity of (2.9), it follows that $i : \mathcal{H} \rightarrow \mathcal{J}$ a morphism in that category.

Diagram (2.9) can be continued with the cokernels of i_U and i_{gU} . We denote $\mathcal{C}(V) = \mathcal{J}(V)/\mathcal{H}(V)$ and put \mathcal{C}^+ for denoting the sheaf associated to the presheaf \mathcal{C} . There are isomorphisms of \mathbb{C} -vector spaces induced canonically between the of vector spaces of the following *commutative* diagram.

$$(2.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}(U) & \xrightarrow{i_U} & \mathcal{J}(U) & \longrightarrow & \mathcal{C}(U) \longrightarrow \mathcal{C}^+(U) \\ & & \alpha|_{U_g} \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}(gU) & \xrightarrow{i_{gU}} & \mathcal{J}(gU) & \longrightarrow & \mathcal{C}(gU) \longrightarrow \mathcal{C}^+(gU) \end{array}$$

The term \mathcal{J}^1 of an injective resolution of \mathcal{H} can be obtained from \mathcal{C}^+ , by embedding \mathcal{C}^+ in an injective object, as we have made with \mathcal{H} . Hence, there exists an isomorphism α^1 of $\mathcal{O}_{G \times X}$ -modules, making commutative the following diagram, where we put \mathcal{J}^0 for the preceding \mathcal{J} .

$$(2.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & b^* \mathcal{H} & \xrightarrow{b^* i} & b^* \mathcal{J}^0 & \xrightarrow{b^* \partial^0} & b^* \mathcal{J}^1 \\ & & \alpha \downarrow & & \alpha^0 \downarrow & & \downarrow \alpha^1 \\ 0 & \longrightarrow & \mu^* \mathcal{H} & \xrightarrow{\mu^* i} & \mu^* \mathcal{J}^0 & \xrightarrow{\mu^* \partial^0} & \mu^* \mathcal{J}^1 \end{array}$$

Continuing the process, we obtain a complex \mathcal{J}^\bullet which satisfies the Lemma. \square

Proposition 8. *If (\mathcal{H}, α) is a G -equivariant \mathcal{O}_X -module, then for each i the cohomology group $H^i(X; \mathcal{H})$ supports a representation of G induced by the isomorphism α .*

Proof. Let \mathcal{J}^\bullet the injective resolution of \mathcal{H} constructed in Lemma 7. As \mathcal{J}^i is G -equivariant, by Proposition 6, the space the $\mathcal{J}^i(X)$ carries the representation ρ^i of G defined by $\rho_g^i = \alpha^i|_{X_g}$. Since the diagrams

$$(2.12) \quad \begin{array}{ccc} \mathcal{J}^i(X) & \xrightarrow{d^i} & \mathcal{J}^{i+1}(X) \\ \rho_g^i \downarrow & & \downarrow \rho_g^{i+1} \\ \mathcal{J}^i(X) & \xrightarrow{d^i} & \mathcal{J}^{i+1}(X) \end{array}$$

are commutative, one has a representation of G on each cohomology group $h^i(\mathcal{J}^\bullet(X))$ of the complex $\mathcal{J}^\bullet(X)$. As \mathcal{J}^\bullet is an injective resolution of the \mathcal{O}_X -module \mathcal{H} and $H^i(X, \mathcal{H})$ is by definition the cohomology group $h^i(\Gamma(X, \mathcal{J}^\bullet))$, the proposition follows. \square

The arguments given in the proof of Proposition 8 are also valid when X is substituted by a G -invariant open subset $S \xrightarrow{j} X$, thus $H^i(S, \mathcal{H})$ also carries a representation of G .

Proof of Theorem 1. As we said, let $j_!$ for the corresponding functor direct image with compact support. The theorem follows from the above observation together with the following identities

$$H^i(S, \mathcal{H}) = R^i\Gamma(S, \mathcal{H}) = \text{Ext}_{\mathcal{O}_X}^i(j_!(\mathcal{O}_S), \mathcal{H}).$$

□

Let $(\mathcal{F}, \gamma), (\mathcal{G}, \beta)$ be G -equivariant \mathcal{O}_X -modules. By Proposition 6, $\mathcal{F}(X)$ and $\mathcal{G}(X)$ support representations of G . We put $\mathcal{K} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the sheaf of homomorphisms from \mathcal{F} to \mathcal{G} . Given an open subset $U \subset X$, for $\Phi_U \in \mathcal{K}(U)$ and $g \in G$, we define $g \cdot \Phi_U \in \mathcal{K}(gU)$ as follows:

Given $\sigma \in \mathcal{F}(gU)$, then $g^{-1} \cdot \sigma \in \mathcal{F}(U)$, with the notation introduced in Proposition 4. Then $\Phi_U(g^{-1} \cdot \sigma) \in \mathcal{G}(U)$ and we put

$$(2.13) \quad (g \cdot \Phi_U)(\sigma) := g \cdot (\Phi(g^{-1} \cdot \sigma)).$$

So, we have constructed an isomorphism

$$\eta|_{U_g} : \mathcal{K}(U) \longrightarrow \mathcal{K}(gU), \quad \Phi_U \mapsto g \cdot \Phi_U.$$

Moreover,

$$\eta|_{(gU)_h} \circ \eta|_{U_g} = \eta|_{U_{hg}}.$$

Therefore, the isomorphisms $\{\eta|_{X_g}\}_g$ define a representation of G on the space $\mathcal{K}(X)$. That is,

Proposition 9. *Let $(\mathcal{F}, \gamma), (\mathcal{G}, \beta)$ be G -equivariant \mathcal{O}_X -modules, then $\mathcal{K}(X)$, with $\mathcal{K} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, supports a representation of G induced by the isomorphisms γ and β .*

2.3. Equivariant complexes. The results of Subsection 2.2 can be generalized to case when \mathcal{F} and \mathcal{G} are G -equivariant objects of the category $\mathbf{D}(\mathcal{O}_X)$, the derived category of coherent sheaves on X . As we said, a G -equivariant object of $\mathbf{D}(\mathcal{O}_X)$ is a pair (\mathcal{A}, α) consisting of an object \mathcal{A} of $\mathbf{D}(X)$ and an isomorphism in the derived category of $\mathcal{O}_{G \times X}$ -modules, which satisfies (2.2).

Given $g \in G$, and open subset U of X , we denote by L_g the diffeomorphism

$$L_g : (g, x) \in U_g \mapsto (g, gx) \in (gU)_g.$$

Thus, $b \circ L_g = \mu : U_g \rightarrow gU$. As L_g is a diffeomorphism the functors L_g^* and $(L_g^{-1})_*$ are the same, one has the following relations among objects

of the derived category $D(\mathfrak{Vect})$, when (\mathcal{A}, α) is an equivariant object.

$$\begin{aligned} R\Gamma(U, \mathcal{A}) &= R\Gamma(U_g, b^* \mathcal{A}) \simeq R\Gamma(U_g, \mu^* \mathcal{A}) = R\Gamma(U_g, L_g^* b^* \mathcal{A}) \\ &= R\Gamma(U_g, R(L_g^{-1})_* b^* \mathcal{A}) = R\Gamma((gU)_g, b^* \mathcal{A}) = R\Gamma(gU, \mathcal{A}). \end{aligned}$$

That is, we have an isomorphism $R\Gamma(U, \mathcal{A}) \xrightarrow{\sim} R\Gamma(gU, \mathcal{A})$. In particular, when $U = X$, for any i there is an isomorphism

$$R\Gamma(X, \mathcal{A}) \xrightarrow{\hat{r}_g} R\Gamma(X, \mathcal{A}).$$

By the cocycle condition $\hat{r}_{hg} = \hat{r}_h \circ \hat{r}_g$.

On the other hand, as α is an isomorphism between two complexes, the α^i 's intertwine with the boundary operators; so, the representation \hat{r} induces a representation r on each space $R^i\Gamma(X, \mathcal{A})$. Since

$$R^i\Gamma(X, \mathcal{A}) = Ext^i(\mathcal{O}_X, \mathcal{A}) = H^i(X, \mathcal{A}),$$

$H^i(X, \mathcal{A})$ carries a representation of G induced by α . This result is a generalization of Proposition 6.

G -equivariant Horseshoe lemma. The well-known Horseshoe lemma (see [27, page 349], [31, page 37]) admits a G -equivariant version, which can be proved following the steps of the proof for the no-equivariant case. This equivariant version can be formulated as follows:

Let (\mathcal{H}', α') , (\mathcal{H}, α) and $(\mathcal{H}'', \alpha'')$ be G -equivariant \mathcal{O}_X -modules and

$$0 \rightarrow \mathcal{H}' \rightarrow \mathcal{H} \rightarrow \mathcal{H}'' \rightarrow 0$$

be a short exact sequence in the category $\text{Coh}^G(X)$. If $\mathcal{H}' \rightarrow \mathcal{J}'^\bullet$ and $\mathcal{H}'' \rightarrow \mathcal{J}''^\bullet$ are resolutions of \mathcal{H}' and \mathcal{H}'' (resp.) in $\text{Coh}^G(X)$ consisting of injective \mathcal{O}_X -modules. Then there are a resolution \mathcal{J}^\bullet of (\mathcal{H}, α) in $\text{Coh}^G(X)$, formed by injective \mathcal{O}_X -modules, and morphisms between the resolutions such that

$$0 \rightarrow \mathcal{J}' \rightarrow \mathcal{J} \rightarrow \mathcal{J}'' \rightarrow 0$$

is an exact sequence of complexes in $\text{Coh}^G(X)$.

Proof of Theorem 2. Since $(\mathcal{G}, \beta) = (\{\mathcal{G}^\bullet\}, \{\beta^\bullet\})$ is a G -equivariant object of $\mathbf{D}(X)$, the \mathcal{O}_X -module \mathcal{G}^j together with β^j is a G -equivariant \mathcal{O}_X -module. We denote by $\partial^j : \mathcal{G}^j \rightarrow \mathcal{G}^{j+1}$ the corresponding boundary operator, as b^* and μ^* are exact functors, $\ker \partial^j$, $\text{im } \partial^j$ and the cohomology $h^j(\mathcal{G}^\bullet)$ are G -equivariant \mathcal{O}_X -modules.

According to Lemma 7, there are resolutions in $\text{Coh}^G(X)$ consisting of injective \mathcal{O}_X -modules for $h^j(\mathcal{G}^\bullet)$ and $\text{im } \partial^{j-1}$. By the G -equivariant Horseshoe lemma applied to the exact sequence

$$0 \rightarrow \text{im } \partial^{j-1} \rightarrow \ker \partial^j \rightarrow h^j(\mathcal{G}^\bullet) \rightarrow 0,$$

there is a resolution for $\ker \partial^j$ satisfying the properties above stated. A new application of the equivariant Horseshoe lemma to the exact sequence

$$0 \rightarrow \ker \partial^j \rightarrow \mathcal{G}^j \rightarrow \operatorname{im} \partial^j \rightarrow 0,$$

permits the construction of a G -equivariant Cartan-Eilenberg resolution $\mathcal{J}^{\bullet, \bullet}$ of the complex \mathcal{G}^\bullet , in which each term is an injective G -equivariant \mathcal{O}_X -module (see [27, Theorem 10.45]).

The isomorphism between the complexes $b^* \mathcal{F}$ and $\mu^* \mathcal{F}$ implies that the representations on $\mathcal{F}^a(X)$ and on $\mathcal{F}^{a+1}(X)$ satisfies $\partial^a(g \cdot \sigma) = g \cdot (\partial^a \sigma)$, for all $\sigma \in \mathcal{F}^a(X)$. The total complex $\mathcal{I} = \operatorname{Tot}(\mathcal{J}^{\bullet, \bullet})$ is a complex in $\operatorname{Coh}^G(X)$ quasi-isomorphic to \mathcal{G} and it is formed by injective \mathcal{O}_X -modules. Thus, by Proposition 6, each vector space $\mathcal{I}^i(X)$ carries a representation of G , which also intertwine with the boundary operators of $\mathcal{I}^\bullet(X)$.

Since (\mathcal{F}, γ) is a G -equivariant object of $\mathbf{D}(X)$, from the argument before Proposition 9, it follows that $\operatorname{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^b(X))$ supports a representation of G . So,

$$(2.14) \quad \mathcal{C}^n := \prod_a \operatorname{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^{a+n}(X)).$$

carries also a representation ρ of G . For \mathcal{C}^\bullet one defines the following boundary operator [19, page 17]

$$(2.15) \quad \delta^n(f_a) = (\partial^{a+n} \circ f_a + (-1)^{n+1} f_{a+1} \circ \partial^a),$$

where,

$$f_a \in \operatorname{Hom}_{\mathcal{O}_X(X)}(\mathcal{F}^a(X), \mathcal{I}^{a+n}(X)).$$

On the other hand, by (2.13), the action of $g \in G$ on f_a is given by $g \cdot (f_a)(\sigma) = g \cdot f_a(g^{-1} \cdot \sigma)$, for all $\sigma \in \mathcal{F}^a(X)$. Since

$$\partial^{a+n} \circ (g \cdot f_a(g^{-1} \sigma)) = g \cdot (\partial^{a+n}(f_a(g^{-1} \cdot \sigma))), \quad \partial^a(g^{-1} \cdot \sigma) = g^{-1} \cdot \partial^a(\sigma),$$

then $\delta^n((g \cdot (f_a))) = g \cdot \delta^n(f_a)$. Hence, ρ induces a representation r of G on the cohomology of \mathcal{C}^\bullet .

On the other hand, the functor

$$R\operatorname{Hom}(\cdot, \cdot) : \mathbf{D}(X) \times \mathbf{D}(X) \rightarrow D(\mathfrak{Vect})$$

assigns to the pair $(\mathcal{F}, \mathcal{G})$ the object represented by the complex \mathcal{C}^\bullet . As $\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) = h^i(R\operatorname{Hom}(\mathcal{F}, \mathcal{G}))$, the representation r is the one claimed in the statement of the theorem. \square

2.4. Equivariant charges. The charge of a brane \mathcal{F} [1, 18, 25] is an element of the cohomology of X defined from certain characteristic classes of X and \mathcal{F} . For a G -equivariant brane, it is natural to define a G -equivariant charge through the respective G -equivariant characteristic classes. The resulting charge will be an element of the equivariant cohomology $H_G(X)$.

Given a G -equivariant brane $\mathcal{F} \in \mathbf{D}(X)$, the complex \mathcal{F}^\bullet is quasi-isomorphic to a complex \mathcal{E}^\bullet consisting of locally free sheaves; that is, \mathcal{E}^i is the sheaf of sections of a G -equivariant vector bundle \mathcal{V}^i .

Each \mathcal{V}^i has the corresponding G -equivariant Chern character $\text{ch}^G(\mathcal{V}^i)$ (see [4, page 212]). We put $\text{ch}^G(\mathcal{E}^\bullet)$ for denoting the $\sum_i (-1)^i \text{ch}^G(\mathcal{V}^i)$. On the other hand, one can consider the G -equivariant \hat{A} -class of X , which will be denoted by $\hat{A}^G(X)$. The equivariant charge of the equivariant brane \mathcal{F} can be defined by the formula

$$(2.16) \quad Q^G(\mathcal{F}) := \text{ch}^G(\mathcal{E}^\bullet) \hat{A}^G(X).$$

Taking into account the relation between the \hat{A} -roof class and the Todd class [23, page 231] the above definition coincides, when G is the trivial group, with the one given in [1].

In some particular cases, the preceding definition has a natural interpretation in terms of the index of an elliptic operator. The exterior bundle $\Lambda^* T^* X$ of X with the connection induced by the Levi-Civita connection and the standard Clifford multiplication is a Dirac bundle (see [23, page 114]). This Dirac bundle has associated the corresponding Dirac operator D^X . If G acts as a group of isometries of X , then D^X is a G -operator [23, page 211]; i.e. D^X is G -equivariant.

Let us assume that the complex \mathcal{E}^\bullet consists of only one nonzero element, \mathcal{E}^0 . The compactness of G allows us to average over the group for obtaining G -invariant metrics and G -invariant connections on \mathcal{V}^0 . On the other hand, the tensor product of $(\Lambda^* T^* X) \otimes \mathcal{V}^0$ is a Dirac bundle (see [23, page 122]) and the corresponding Dirac operator D is also G -equivariant, by the G -invariance of the metric and the connection. Since D is elliptic, if X is compact, $\ker D$ and $\text{coker } D$ are representations of G of finite dimension. Denoting by $R(G)$ the character ring of G , the G -index of D , $\text{ind}_G(D)$, is an element of $R(G)$. For $g \in G$ the virtual character $\chi(D)(g)$ of D at g is defined by

$$\chi(D)(g) = \text{trace}(g|_{\ker D}) - \text{trace}(g|_{\text{coker } D}).$$

The equivariant index theorem [4, Chapter 8], asserts that in a neighborhood of $0 \in \mathfrak{g} := \text{Lie}(G)$

$$(2.17) \quad \chi(D) \circ \exp = \int_X Q^G(\mathcal{E}^0).$$

The value of $\chi(D)(\exp(\xi))$, for $\xi \in \mathfrak{g}$, can be calculated by the localization formula in equivariant cohomology. The result is the Atiyah-Segal-Singer fixed point formula [4, 23].

3. PARTICULAR CASES.

In this section, we show the form that the results of Section 2 adopt in some simple cases.

A consequence of the Grothendieck spectral sequence [13, page 207] is the known Local-to-Global Ext spectral sequence, which allows to determine the Ext groups from the sheaves $\mathcal{E}xt$. Given the \mathcal{O}_X -modules \mathcal{F} and \mathcal{G} , the first quadrant spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}))$$

abuts to $Ext^n(\mathcal{F}, \mathcal{G})$.

Let \mathcal{F} be a locally free \mathcal{O}_X -module of finite rank, then $0 \rightarrow \mathcal{F} \xrightarrow{1} \mathcal{F} \rightarrow 0$ is a resolution of \mathcal{F} , which can be used to determine the sheaves $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G})$ [17, Proposition 6.5, page 234]. So, the sheaves $\mathcal{E}xt$ are the cohomology of the trivial complex consisting of the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ at the position 0 and zeros in the other positions. Thus,

$$\mathcal{E}xt^0(\mathcal{F}, \mathcal{G}) = \mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

and $\mathcal{E}xt^q(\mathcal{F}, \mathcal{G}) = 0$, for $q \neq 0$. Therefore, the spectral sequence degenerates at the second page and

$$Ext^p(\mathcal{F}, \mathcal{G}) = H^p(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})).$$

Thus, we have the following proposition.

Proposition 10. *Let \mathcal{F} be a locally free sheaf of finite rank on X , then for any coherent sheaf \mathcal{G} ,*

$$St^q(\mathcal{F}, \mathcal{G}) = H^q(X, \mathcal{F}^\vee \otimes \mathcal{G}),$$

where \mathcal{F}^\vee is the dual sheaf of \mathcal{F} .

Given an invertible sheaf \mathcal{G} , i.e. a rank 1 locally free \mathcal{O}_X -module, we put \mathcal{G}^n for denoting the tensor product $\mathcal{G}^{\otimes n}$. The following corollary is a consequence from the proposition together with the fact that \mathcal{G} is ample (see [17, Proposition 5.3, page 229]).

Corollary 11. *Let \mathcal{F} be a locally free sheaf of finite rank and \mathcal{G} an ample invertible sheaf, then there is an integer n_0 such that*

$$St^q(\mathcal{F}, \mathcal{G}^n) = 0,$$

for all $q > 0$ and all $n > n_0$

Let \mathcal{V} and \mathcal{W} be holomorphic vector bundles over X with finite rank. We put $\mathcal{F} := \mathcal{O}(\mathcal{V})$ and $\mathcal{G} := \mathcal{O}(\mathcal{W})$ for denoting the respective sheaves of holomorphic sections. Then \mathcal{F} is a locally free sheaf and by the Proposition 10,

$$St^p(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W})) = H^p(X, \mathcal{O}(\mathcal{V}^\vee \otimes \mathcal{W})),$$

where \mathcal{V}^\vee is the dual vector bundle of \mathcal{V} .

Let us assume that \mathcal{V} and \mathcal{W} are G -equivariant vector bundles on X , with G a compact Lie group. We denote by χ_V and χ_W the characters of the corresponding representations on the spaces of sections. Then $St^0(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W}))$ supports the representation with character $\chi := \bar{\chi}_V \chi_W$. If we write $\chi = \sum_k n_k \chi_k$, where the χ_k are characters of a complete family of irreducible representations of G , then the open string states with ghost number 0, between two branes wrapped on the whole X with gauge bundles \mathcal{V} and \mathcal{W} , can be expressed as the following sum direct of G -invariant subspaces

$$St^0(\mathcal{O}(\mathcal{V}), \mathcal{O}(\mathcal{W})) \simeq \bigoplus_k n_k B_k,$$

where B_k is a subspace on which the representation of G has character χ_k . The natural number n_k before the subspace B_k means the direct sum of n_k summands equal to B_k .

3.1. Flag manifolds. When X is a flag manifold of semisimple group, the result stated in Theorem 3 admits, for certain spaces of strings, a more precise formulation derived from the Borel-Bott-Weil theorem (see [7, 32], for a brief exposition [22, pages 13-22]).

We remind some basic facts about flag manifolds; for details see [8]. Let \mathfrak{g} be the Lie algebra of a linear connected semisimple complex Lie group $G_{\mathbb{C}}$. We assume that a Cartan subalgebra \mathfrak{h} of \mathfrak{g} has been fixed. If \mathfrak{v} , an $\text{ad}(\mathfrak{h})$ -invariant subspace of \mathfrak{g} , then the set roots of \mathfrak{h} in \mathfrak{v} will be denoted by $\Delta(\mathfrak{v})$. Given a system $\Delta^+ \subset \Delta(\mathfrak{g})$ of positive roots, the parabolic subalgebras of \mathfrak{g} can be constructed as follows: If Γ a set of simple positive roots, we put $\Delta(\Gamma) = \text{Span}_{\mathbb{Z}}(\Gamma) \cap \Delta(\mathfrak{g})$. The set Γ determines the parabolic subalgebra $\mathfrak{p} := \mathfrak{l} \oplus \mathfrak{u}$, where

$$(3.1) \quad \mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\Gamma)} \mathfrak{g}_{\alpha}, \quad \text{and} \quad \mathfrak{u} := \bigoplus_{\alpha \in \Delta^+ \setminus \Delta(\Gamma)} \mathfrak{g}_{\alpha}.$$

When $\Gamma = \emptyset$, the corresponding parabolic algebra is the Borel subalgebra determined by Δ^+ .

The parabolic subgroup P associated with the algebra \mathfrak{p} can be expressed $P = L_{\mathbb{C}}U_{\mathbb{C}}$ (Levi decomposition), where $L_{\mathbb{C}} \cap U_{\mathbb{C}} = \{1\}$, $L_{\mathbb{C}}$ is a reductive group and $U_{\mathbb{C}}$ is nilpotent.

Henceforth in this Subsection, we assume that a parabolic subgroup P of $G_{\mathbb{C}}$ has been fixed. As $G_{\mathbb{C}}$ is connected, the normalizer $N_{G_{\mathbb{C}}}(\mathfrak{p})$ coincide with P . Hence, the flag variety $X = G_{\mathbb{C}}/P$ can be identified with the set of all the parabolic subalgebras which are $G_{\mathbb{C}}$ -conjugated to \mathfrak{p} . X is a compact simply connected Kähler manifold (see for example [8, 32]). In [15], Grantcharov showed several examples flag manifolds which are Calabi-Yau.

By $G \subset G_{\mathbb{C}}$ we denote a real form of $G_{\mathbb{C}}$; i.e. a Lie subgroup of $G_{\mathbb{C}}$, such that $\mathfrak{g} = \text{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$. As subgroup of $G_{\mathbb{C}}$, G acts on X and there is only finitely many G -orbits on X . In the case G is a compact real form of $G_{\mathbb{C}}$, the G -action on X is transitive. In [31] there is a detailed exposition of the properties G -action on X , a shorter one can be looked up in [33].

We assume that G is a *compact* real form of $G_{\mathbb{C}}$. We put $L := L_{\mathbb{C}} \cap G$, then transitive action of G on X permits identify X and G/L . Using the above Levi decomposition of P , an irreducible representation r of L on the finite dimensional complex vector space V can be extended to a holomorphic representation of P , in which the action of the factor $U_{\mathbb{C}}$ is trivial. With the P -action on V one can define the following G -homogeneous vector bundle over $X = G_{\mathbb{C}}/P$

$$(3.2) \quad \mathcal{V} = G \times_P V.$$

Let $\lambda \in \mathfrak{h}^*$ be the highest weight of the above representation r of L . Define

$$(3.3) \quad i(\lambda) := \#\{\alpha \in \Delta^+ \mid \langle \lambda + \rho, \alpha \rangle < 0\},$$

ρ being the half sum of the positive roots.

As $St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{V})) = H^i(X, \mathcal{O}(\mathcal{V}))$, by a direct application of Borel-Bott-Weil theorem, we deduce the following proposition about the spaces of string on the flag manifold $X = G_{\mathbb{C}}/P$.

Proposition 12.

- (1) If $\lambda + \rho$ is singular, that is $\langle \lambda + \rho, \alpha \rangle = 0$ for some root $\alpha \in \Delta(\mathfrak{g})$, then $St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{V})) = 0$ for all i .
- (2) If $\lambda + \rho$ is regular, that is $\langle \lambda + \rho, \alpha \rangle \neq 0$ for all $\alpha \in \Delta(\mathfrak{g})$, then there exists w in the Weyl group so that $w(\lambda + \rho)$ is dominant with respect to $\Delta^+ \cap \Delta(\mathfrak{l})$. In this case, $St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{V})) = 0$ for

$i \neq i(\lambda)$ and $St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{V}))$ is the irreducible representation of G of highest weight $w(\lambda + \rho) - \rho$.

As a direct consequence of a vanishing theorem proved in [29], we can state the following result, which yields a upper bound for the ghost number of the strings between two particular types of branes in X ; in other words, a upper bound in the number of nonzero summands of (1.2).

Proposition 13. *Let S be an open G -orbit (G not necessarily compact) in the flag manifold $X = G_{\mathbb{C}}/P$. We denote by $j : S \hookrightarrow X$ the inclusion and by s the complex dimension of a maximal compact subvariety of S . If \mathcal{H} is a coherent sheaf on X , then $St^i(j_!(\mathcal{O}_S), \mathcal{H}) = 0$, for $i > s$. Where $j_!(\mathcal{O}_S)$ is the direct image of \mathcal{O}_S by the inclusion.*

3.2. Toric varieties. Known properties of the cohomology of toric varieties will allow us to express the decomposition (1.2) in a more precise terms, when \mathcal{F} and \mathcal{G} are particular branes on a toric manifold. We will also apply the localization formula in cohomology equivariant to (2.17), when X is a toric manifold.

Let Σ be a fan in $N = \mathbb{Z}^r$, we will denote by X the toric variety defined by Σ [9, 10, 12, 26]. We put $M := Hom_{\mathbb{Z}}(N, \mathbb{Z})$, $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and T for the torus

$$T = N \otimes \mathbb{C}^{\times} = Hom_{\mathbb{Z}}(M, \mathbb{C}^{\times}).$$

Given $m \in M$ we denote by χ^m the homomorphism

$$\chi^m : t \in T \mapsto t(m) \in \mathbb{C}^{\times}.$$

That is, the χ^m 's are the characters of the irreducible representations of T .

We put $\Sigma(1)$ for denoting the set of 1-dimensional cones in Σ , and given $\rho \in \Sigma(1)$, there is a unique primitive element $v_{\rho} \in N \cap \rho$, such that the cone ρ can be expressed as $\mathbb{R}_{\geq 0}v_{\rho}$, and any cone σ of Σ can be written

$$(3.4) \quad \sigma = \sum_{\rho \in \Sigma(1) \cap \sigma} \mathbb{R}_{\geq 0}v_{\rho}.$$

We assume that X is nonsingular. Given a family $(a_{\rho}) \in \mathbb{Z}^{\Sigma(1)}$, we put $\psi(v_{\rho}) = a_{\rho}$, for all $\rho \in \Sigma(1)$. By (3.4), ψ can be extended to a function ψ defined on the support of Σ , which is linear on each cone of Σ . On the other hand, the family (a_{ρ}) determines the following divisor on X

$$(3.5) \quad A = - \sum_{\rho} a_{\rho} V(\rho),$$

where $V(\rho)$ is the closure of the orbit of ρ under the T -action. A is a T -invariant divisor of X , which determines a T -equivariant line bundle \mathcal{L} , in the usual way.

By Proposition 8, $H^i(X, \mathcal{O}(\mathcal{L}))$ supports a representation of T . As $\text{Ext}^i(\mathcal{O}_X, \mathcal{O}(\mathcal{L})) = H^i(X, \mathcal{O}(\mathcal{L}))$, the decomposition of $St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{L}))$ stated in Theorem 3 can be expressed, for this particular case, in terms of the local cohomology of $N_{\mathbb{R}}$. In fact, from Theorem 2.6 of [26] (see also [12, page 74]) we deduce the following proposition.

Proposition 14. *If Σ is a smooth fan and \mathcal{L} is the line T -equivariant bundle associated to the divisor (3.5), then*

$$St^i(\mathcal{O}_X, \mathcal{O}(\mathcal{L})) = \sum_{m \in M} H_{Z(m)}^i(N_{\mathbb{R}}, \mathbb{C}) \chi^m,$$

where $Z(m) = \{v \in N_{\mathbb{R}}, | \langle m, v \rangle \geq \psi(v)\}$.

Let A and A' be T -invariant divisors of X . We denote by ψ and ψ' the corresponding linear support functions associated to A and A' , respectively. We put \mathcal{L} and \mathcal{L}' for the respective line bundles. One says that ψ is strictly convex if $\lambda\psi(u) + (1-\lambda)\psi(v) < \psi(\lambda u + (1-\lambda)v)$, for all $\lambda \in [0, 1]$ and $u, v \in N_{\mathbb{R}}$. A known fact is that $\mathcal{O}(\mathcal{L})$ is ample if ψ is strictly convex [12, page 70]. The following proposition shows the form adopted by Theorem 3, when $\mathcal{F} = \mathcal{O}(\mathcal{L})$ and $\mathcal{G} = \mathcal{O}(\mathcal{L}')$.

Proposition 15. *With the above notations, if $\psi' - \psi$ is strictly convex, then*

$$St^q(\mathcal{O}(\mathcal{L}), \mathcal{O}(\mathcal{L}')) = 0, \quad \text{for } q \neq 0,$$

and

$$St^q(\mathcal{O}(\mathcal{L}), \mathcal{O}(\mathcal{L}')) = \bigoplus_{m \in P} \mathbb{C} \cdot \chi^m,$$

P being $\{m \in M \mid \langle m, n \rangle \geq \psi'(n) - \psi(n), \text{ for all } n \in N_{\mathbb{R}}\}$.

Proof. By Proposition 10, $St^q(\mathcal{O}(\mathcal{L}), \mathcal{O}(\mathcal{L}')) = H^q(X, \mathcal{O}(\mathcal{L}^\vee \otimes \mathcal{L}'))$. As the support function associated to $\mathcal{O}(\mathcal{L}^\vee \otimes \mathcal{L}')$ is $\psi' - \psi$, the proposition follows from Theorem 2.7 and Corollary 2.9 in [26]. \square

In order to apply the fixed point formula to (2.17) when X is a toric manifold, we make two Remarks.

Remark 1. Let us assume that the torus T acts *trivially* on a connected manifold S and that \mathcal{W} is a T -equivariant vector bundle over S with rank m . By the equivariant splitting principle, we can assume that \mathcal{W} is a direct sum of T -equivariant line bundles

$$\mathcal{W} = \bigoplus_{j=1}^m \mathcal{L}_j.$$

The action of T on \mathcal{W} is defined by m weights φ_j and the T -equivariant Chern class of \mathcal{L}_j is given by (see [14, page 317])

$$(3.6) \quad c_1^T(\mathcal{L}_j) = c_1(\mathcal{L}_j) + \frac{1}{2\pi}\varphi_j.$$

The T -equivariant Chern character of \mathcal{W} is (see [23, page 234])

$$(3.7) \quad \text{ch}^T(\mathcal{W}) = \sum_{j=1}^m \exp(c_1^T(\mathcal{L}_j)).$$

Remark 2. Let p be a fixed point for the T -action on the toric manifold associated with the fan $\Sigma \subset \mathbb{R}^n$. We denote by $\nu_{i,p} \in 2\pi(\mathbb{Z})^n$, $i = 1, \dots, n$, the weights of the isotropy representation of T on $T_p X$. The fixed points of the T -action are in bijective correspondence with the n -dimensional cones in Σ [10, §3.2]. If the point p is associated with the cone σ , then

$$(3.8) \quad \omega_{i,p} = \frac{\nu_{i,p}}{2\pi}$$

are the generators of $\sigma^\vee \cap M$, where σ^\vee is the dual cone of σ .

In the statement of the following proposition, \mathcal{V}^0 is the vector bundle introduced in Subsection 2.4.

Proposition 16. *Let X be the toric manifold associated to the fan Σ , denoting by $\{\varphi_{j,p}\}_{j=1,\dots,m}$ the weights of the representation of T on the fibre of \mathcal{V}^0 at p , then (2.17) is equal to*

$$(3.9) \quad 2^{-n} \sum_{p \in X^T} \left(\sum_{j=1}^m e^{\frac{1}{2\pi}\varphi_{j,p}} \right) \prod_{i=1}^n \left(\sinh \left(\frac{\omega_{i,p}}{2} \right) \right)^{-1},$$

where X^T is the set of fixed points of X for the T -action.

Proof. The localization theorem in equivariant cohomology [16, §10.9] allows us to calculate the value (2.17) as a sum of contributions of the connected components of X^T . As X^T is discret, the localization formula adopts the following form

$$(3.10) \quad \chi(D) \circ \exp = (2\pi)^n \sum_{p \in X^T} \frac{Q^G(\mathcal{E}^0)(p)}{\prod_i \nu_{i,p}},$$

where the $\nu_{i,p}$ are the weights of the isotropy representation of T at the fixed point p .

From (3.6) and (3.7), it follows

$$\text{ch}^T(\mathcal{V}^0)|_p = \sum_{j=1}^m e^{\frac{1}{2\pi}\varphi_{j,p}}.$$

Similarly (see [23, page 231]),

$$\hat{A}^T(TX)|_p = \prod_{i=1}^n \frac{\omega_{i,p}}{2} \left(\sinh \left(\frac{\omega_{i,p}}{2} \right) \right)^{-1}.$$

The proposition follows from (2.16) together with (3.8). \square

Note that the contribution of the manifold X to (3.9) is encoded in the n -dimensional cones of the fan Σ .

REFERENCES

- [1] Aspinwall, P. S.: *D-branes on Calabi-Yau manifolds*. In *Progress in String Theory*. Pages 1-152. World Sci. Publ. (2005).
- [2] Aspinwall et al. *Dirichlet branes and mirror symmetry*. Clay mathematics monographs vol 4. Amer. Math. Soc. (2009)
- [3] Aspinwall, P. S., Lawrence, A. E.: *Derived categories and zero-brane stability*. JHEP **08**, 004 (2001).
- [4] Berline, N., Getzler, E., Vergne, M.: *Heat Kernels and Dirac Operators*. Springer-Verlag, (1991).
- [5] Berline, N., Vergne, M.: *The equivariant index and Kirillov character formula*. Amer. J. Math. **107**, 1159-1190 (1985).
- [6] Bernstein, J., Lunts, V.: *Equivariant sheaves and functors*. Lecture Notes in Mathematics 1578, Springer-Verlag, (1994).
- [7] Bolton, V. Schmid, W. : *Discrete series*. Proc. Symposia in Pure Math. **61**, 83-113 (1997).
- [8] Borel, A.: *Linear algebraic groups*. Springer-Verlag, (1991).
- [9] Cox, D. A., Katz, S.: *Mirror symmetry and algebraic geometry*. AMS (1999).
- [10] Cox, D., Little, J., Schenck, H.: *Toric varieties*. AMS (2011).
- [11] Douglas, M. R.: *D-branes, categories and $N = 1$ supersymmetry*. J. Math. Phys **42**, 2818-2843 (2001).
- [12] Fulton, W.: *Introduction to toric varieties*. Princeton University Press (1993).
- [13] Gelfand, S. I., Manin, Y. I.: *Methods of homological algebra*. Springer (2002).
- [14] Ginzburg, V., Guillemin, V., Karshon, Y.: *Moment maps, cobordisms, and Hamiltonian group actions*. Mathematical Surveys and Monographs, volume 98. AMS (2002).
- [15] Grantcharov, G.: *Geometry of compact complex homogeneous spaces with vanishing first Chern class*. Adv. Math. **226**, 3136-3159 (2011).
- [16] Guillemin, V., Sternberg, S.: *Supersymmetry and equivariant de Rham theory*. Springer-Verlag, (1999).
- [17] Hartshorne R.: *Algebraic geometry*. Springer-Verlag (1983).
- [18] Harvey, J. A.: *TASI 2003 Lectures on anomalies*. 2005.
- [19] Iversen, B.: *Cohomology of sheaves*. Springer (1986).
- [20] Kashiwara, M., Schapira, P.: *Sheaves on manifolds*. Springer-Verlag, (2002).
- [21] Katz, S., Sharpe, E.: *D-branes, open string vertex operators, and Ext groups*. Adv. Theor. Phys. **6**, 979-1030 (2003).
- [22] Knapp, A. W., Vogan, D. A.: *Cohomological induction and unitary representations*. Princeton U.P. (1995).
- [23] Lawson, H. B. , Michelson, M.: *Spin Geometry*. Princeton (1989).
- [24] Mac Lane, S.: *Homology*. Springer-Verlag (1975).

- [25] Minasian, R., Moore, G.: K-theory and Ramond-Ramond charge. J High Energy Phys. **11**, (1997) 002.
- [26] Oda, T.: Convex bodies and algebraic geometry. Springer-Verlag (1988).
- [27] Rotman, J. J.: An introduction to homological algebra. Springer (2008).
- [28] Sharpe, E.: D -branes, derived categories and Grothendieck groups. Nucl. Physics **B561**, 433-450 (1999).
- [29] Schmid, W., Wolf, J.A.: A vanishing theorem for open orbits on a complex flag manifold. Proc. Amer. Soc. **92**, 461-464 (1984).
- [30] Weibel, Ch. A.: An introduction to homological algebra. Cambridge U.P. (1997).
- [31] Wolf, J. A.: The action of real semisimple group on a complex flag manifold I. Orbit structure and holomorphic arc components. Bull. Amer. Aath. Soc. **75**, 1121-1237 (1969).
- [32] Wolf, J. A.: Flag manifolds and representation theory. In Geometry and representation theory of real and p -adic Lie groups. Birkäuser (1997), pages 273-323.
- [33] Zierau, R.: Representatons in Dolbeault cohomology. In Representation Theory of Lie Groups. (J. Adams and D. Vogan editors). IAS/Park City Mathematics Series, vol 8. AMS (2000) pp. 91-146.

DEPARTAMENTO DE FÍSICA. UNIVERSIDAD DE OVIEDO. AVDA CALVO SOTELO.
33007 OVIEDO. SPAIN.

E-mail address: vina@uniovi.es